UNCLASSIFIED

AD NUMBER

ADB279872

NEW LIMITATION CHANGE

TO

Approved for public release, distribution unlimited

FROM

Distribution: Further dissemination only as directed by OSRD, Washington, DC 20330, Jun 1945; or higher DoD authority.

AUTHORITY

AFOSR, per DTIC Form 55, dtd 14 Nov 2002

THIS PAGE IS UNCLASSIFIED

.9646 a

~~[illegible]~~ aPG(c) 3349/1337

C ~~ONFIDENTIAL~~

28 Jun 45

Division 2, National Defense Research Committee
of the
Office of Scientific Research and Development

PENETRATION THEORY: SEPARABLE FORCE LAWS AND THE TIME OF PENETRATION

by

Richard A. Beth

NDRC Report No. A-333
OSRD Report No. 5258

Copy No. 53

C ONF ~~IDENTIAL~~

[redacted] UNCLASSIFIED

Division 2, National Defense Research Committee
of the
Office of Scientific Research and Development

PENETRATION THEORY: SEPARABLE FORCE LAWS AND THE TIME OF PENETRATION

by

Richard A. Beth

NDRC Report No. A-333
OSRD Report No. 5258

ted by

Walker Bleakney

Walker Bleakney
Princeton University

Approved on June 28, 1945
for submission to the Committee

Ralph J. Slutz

Ralph J. Slutz, Technical Aide
Division 2
Effects of Impact and Explosion

[redacted]

UNCLASSIFIED

The NDRC Technical Reports Section
for armor and ordnance edited
this report and prepared it for duplication.

Requests for additional reports should be
addressed through liaison channels to:
National Defense Research Committee
Technical Reports Section
Room 4724, Munitions Building
Washington 25, D.C.

Office of Scientific Research and Development

NATIONAL DEFENSE RESEARCH COMMITTEE

Technical Reports Section

Room 4724, Munitions Building

Washington 25, D.C.

July 21, 1945

CONFIDENTIAL

TO: Recipients of NDRC Report A-333 (OSRD-5258), Penetration theory: Separable force laws and the time of penetration

FROM: Technical Reports Section

The second equation on page 1 is incorrect. It should read

$$R = a(x)v^{2\lambda} + b(x)v^2$$

instead of

$$R = a(x)v^2\lambda + b(x)v^2.$$

Please make this correction on your copy.

Preface

The work described in this report is pertinent to the project designated by the War Department Liaison Officer as CE-36 and to the project designated by the Navy Department Liaison Officer as NO-12. The report constitutes a progress report under Contract OEMsr-260 with Princeton University.

Initial distribution of copies

Nos. 1 to 21 to Liaison Office, OSRD, for
- Admiralty (Operational Research Section)
- Air Ministry (Scientific Advisor; Director Armament Requirements)
- Department of Scientific and Industrial Research (Road Research Laboratory; National Physical Laboratory)
- Ministry of Aircraft Production (Director Armament Development; Central Library)
- Ministry of Home Security (Research and Experiments Department)
- Ministry of Supply (Armament Research Department; Armament Design Department; Anti-Concrete Committee; SR 7)
- Ordnance Board
- War Office (Scientific Advisor; Army Operational Research Group)
- Additional distribution (6 copies)

Nos. 22 to 28 to Office of the Executive Secretary, OSRD;

No. 29 to Office of Field Service, OSRD (V. V. Sides);

No. 30 to J. P. Baxter, III, Historian, OSRD;

No. 31 to R. C. Tolman, Vice Chairman, NDRC;

Nos. 32 and 33 to E. B. Wilson, Jr., Chief, Division 2;

No. 34 to W. Bleakney, Deputy Chief, Division 2;

No. 35 to H. L. Bowman, Member, Division 2;

No. 36 to W. E. Lawson, Member, Division 2;

No. 37 to D. P. MacDougall, Member, Division 2;

No. 38 to J. von Neumann, Member, Division 2;

No. 39 to S. A. Vincent, Member, Division 2;

No. 40 to M. P. White, Technical Aide, Division 2;

No. 41 to R. A. Beth, Consultant, Division 2;

No. 42 to P. W. Bridgman, Consultant, Division 2;

No. 43 to C. W. Curtis, Consultant, Division 2;

No. 44 to N. M. Newmark, Consultant, Division 2;

No. 45 to A. H. Taub, Consultant, Division 2;

Nos. 46 and 47 to Division 2 Library, Princeton University;

No. 48 to L. H. Adams, Chief, Division 1;

No. 49 to H. B. Allen, Deputy Chief, Division 1;

No. 50 to Mina Rees, Technical Aide, Applied Mathematics Panel;

Nos. 51 to 53 to Aberdeen Proving Ground (Ballistic Research Laboratory; Maj. J. S. Lieb; O. Veblen);

Nos. 54 and 55 to Air Technical Service Command, Wright Field, Dayton, Ohio (Maj. J. P. AuWerter; Capt. R. T. Franzel);

Nos. 56 to 57 to Army Air Forces (Lt. Col. J. M. Gruitch; Th. von Kármán, Scientific Advisory Group);

No. 58 to Army Ground Forces (Col. H. R. Cox);

Nos. 59 and 60 to Corps of Engineers (Lt. Col. W. J. New);

No. 61 to Frankford Arsenal (Lt. Col. C. H. Greenall);

No. 62 to Joint Target Group;

Nos. 63 to 65 to Ordnance Department (Col. S. B. Ritchie; Lt. Col. P. L. Christensen; S. Feltman);

No. 66 to War Department Liaison Officer with NDRC for transmittal to Col. P. Schwartz, Air Ordnance Officer, European Theater;

No. 67 to Watertown Arsenal (Director of Laboratory);

No. 68 to Watervliet Arsenal (Col. S. L. Conner);

No. 69 to Bureau of Aeronautics (Comdr. E. Tatom);

Nos. 70 to 73 to Bureau of Ordnance [Research and Development Division, Re3 (2 copies); Comdr. W. A. Walter; A. Wertheimer];

Nos. 74 and 75 to Bureau of Ships (Comdr. J. M. Fluke; Comdr. C. H. Gerlach);

No. 76 to Bureau of Yards and Docks (Design Manager);

No. 77 to Commander in Chief, U.S. Fleet (Group M);

No. 78 to David Taylor Model Basin (Capt. W. P. Roop);

No. 79 to Joint Intelligence Center, Pacific Ocean Area (Lt. Comdr. T. C. Wilson, Target Analysis Section);

No. 80 to U.S. Naval Amphibious Training Base, Fort Pierce, Fla. (Demolition Research Unit);

No. 81 to U.S. Naval Gun Fire Liaison School, Amphibious Training Base, San Diego, Calif. (Officer in Charge);

Nos. 82 and 83 to U.S. Naval Proving Ground, Dahlgren, Va. (Experimental Officer; Lt. Comdr. R. A. Sawyer);

No. 84 to U.S. Naval Research Laboratory (R. Gunn);

No. 85 to R. J. Emrich, Princeton University;

No. 86 to I. M. Freeman, Princeton University;

No. 87 to C. W. Lampson, Princeton University;

No. 88 to J. G. Stipe, Jr., Princeton University.

CONTENTS

		Page
Abstract		1
Section		
1.	Introduction	2
2.	Notation	2
3.	Equation of motion and time of penetration	3
4.	Classification of penetration theories	3
5.	The general separable force law $R = c \cdot g(x) \cdot f(v)$	4
6.	The force law $R = cx^a v^b$	9
7.	The Poncelet force law $R = a + bv^2$	13
8.	Sectional-pressure theories and the time of penetration	14
9.	A perforation hypothesis	16
10.	Conclusions	17
List of references		18

List of Figures

Figure		Page
1.	Classification of penetration theories	5
2.	Graphical representation of Eqs. (14) and (15) for a separable force law	6
3.	$K = \frac{v_0 t_1}{x_1}$ for the separable force law $R = cx^a v^b$	11
4.	$K = \frac{v_0 t_1}{x_1}$ for the Poncelet force law	12

PENETRATION THEORY: SEPARABLE FORCE LAWS AND THE TIME OF PENETRATION

Abstract

The equation of motion of a penetrating projectile can be integrated if the force law depends only on the depth of penetration x and the velocity v, and has either of the two general forms:

$$R = c \cdot g(x) \cdot f(v)$$

or

$$R = a(x)\, v^{2\lambda} + b(x)\, v^2,$$

both of which include as special cases the classical sectional-pressure theories of penetration. The second of these has been discussed in a previous report. The present report summarizes the separable force laws represented by the first.

For a separable force law the relation between x and v during penetration can be expressed in terms of the striking velocity v_0 and the maximum penetration x_1 in such a way that the mass of the projectile and the target strength parameter c are eliminated. This leads to simplified formulae for computing times of penetration. The formulae for a number of special cases of separable force laws are tabulated.

The separable force law $R = cx^a v^b$ leads to a penetration formula of the form $x_1 = \text{constant} \times v_0^{\alpha}$, where $\alpha = (2-b)/(1+a)$. It is shown that $K = v_0 t_1/x_1$, where t_1 is the total time of penetration, depends only on the constants a and b, and a graph is given for finding the constant K from assumed values for a and b.

For the Poncelet force law, $R = a + bv^2$, K is a function of $u = v\sqrt{b/a}$ only, and a graph is given for finding K.

Examples are given to illustrate the use of the graphs.

It is shown for any sectional-pressure theory of penetration (that is, $g(x) \equiv 1$) that (a) the remaining penetration $x_1 - x$ beyond any point x is related to the remaining velocity v in the same way that x_1 is related to v_0, and (b) the remaining time of penetration $t_1 - t$ is related to v in the same way that t_1 is related to v_0. It follows that $v_0 t_1/x_1 = K(v_0)$ and $v(t_1 - t)/(x_1 - x) = K(v)$; thus both t_1 and t can be deduced with the aid of a graph of $K(v)$ for the particular sectional-pressure theory assumed. This is illustrated by examples.

A separable force law for perforation leads to a relation between limit, striking, and residual velocities of the form $F(v_\ell) = F(v_0) - F(v_r)$, which is independent of the projectile mass and the target strength coefficient c.

In conclusion, it is suggested that sectional-pressure theories tend to give too large estimates for t_1 and that the simple assumption $v_0 t_1/x_1 = 2$ will often give better results.

1. Introduction

The need for a method of estimating forces, velocities, and times during penetration in concrete and other materials has been discussed in previous reports.[1/] Direct measurements of these quantities are needed but are not as yet available.[2/] Further work[3/] has been done to explore the possibility of making reasonable estimates by indirect theoretical methods. These depend on finding a suitable approximation to the force law governing the resistance offered by the target to the projectile during penetration. While this work is specifically directed toward the problem of concrete penetration, the possibility of applying similar methods to other materials is to be kept in mind.

2. Notation

The following notation will be used.

Projectile parameters assumed constant during penetration.	w = weight. A = maximum cross-sectional area. $P = w/A$ = sectional pressure. $P' = P/g$ = sectional density, where g is the acceleration due to gravity.
Kinematic variables during penetration.	x = depth of nose penetration. t = time from beginning of penetration. $v = dx/dt$ = remaining velocity.
Initial conditions at impact.	$x = 0$. $t = 0$. $v = v_o$ = striking velocity.
Final conditions at maximum penetration.	$x = x_1$ = maximum penetration. $t = t_1$ = time of penetration. $v = 0$.

1/ Introduction in Ref. 2 and Introduction in Ref. 4. See List of References at the back of this report.

2/ Reference 4 describes the development of an experimental method for making such measurements on materials like concrete.

3/ See previous work in Ref. 3.

3. Equation of motion and time of penetration

Using the notation given in Sec. 2, the equation of motion of a projectile during penetration is

$$P' \frac{dv}{dt} = P' v \frac{dv}{dx} = -R, \tag{1}$$

where R, the resisting pressure, is the instantaneous value of the resisting force divided by A, the maximum cross-sectional area of the projectile.

Times may be computed from either of the two expressions

$$t = \int_0^x \frac{dx}{v}, \tag{2}$$

$$t_1 - t = P' \int_0^v \frac{dv}{R}, \tag{3}$$

provided either that v can be expressed as a function of x in Eq. (2) or that R can be expressed as a function of v in Eq. (3).

4. Classification of penetration theories

The resisting pressure R can depend on certain target and projectile parameters that remain constant during the motion and it can depend on certain quantities, like x and v, that vary during the motion. It is assumed that R does not depend on the mass of the projectile.

For the theoretical treatment of penetration it is commonly assumed that R depends only on the variables x and v although it may actually depend on more complicated variables during the motion.[4/]

The force laws corresponding to certain of the classical theories of penetration are:

(4) Robins-Euler: $R = c$,

(5) Poncelet: $R = a + bv^2$,

(6) Pétry: $R = a\left(1 + \frac{v^2}{215000}\right)$, with v expressed in feet per second,

---

4/ See page 9 in Ref. 2.

to which may be added

(7) $$R = cv^b.$$

With $b = 0.68$ the latter reproduces de Giorgi's values[5/] for masonry and concrete very well. Some attention has been devoted to Eq. (7) during the present war both here and in England.[6/]

In these cases the equation of motion, Eq. (1), can be integrated explicitly. In fact, they are all special cases of either of the following two much more general force laws for which Eq. (1) can be integrated:

(8) Separable: $R = c \cdot g(x) \cdot f(v)$,

(9) Generalized Poncelet: $R = a(x)v^{2\lambda} + b(x)v^2$.

Figure 1 gives a schematic classification of some of the theories of penetration obtained by progressive specialization (indicated by the arrows) of these two general force laws. Further specializations of Eq. (9) could easily be added to the diagram.

A treatment of the generalized Poncelet theory based on the force law given by Eq. (9) has been given in a previous report.[7/] The consequences of assuming a separable force law, Eq. (8), have also been treated[8/] but will be discussed further in this report.

5. The general separable force law $R = c \cdot g(x) \cdot f(v)$

With the separable force law, Eq. (8), $g(x) \cdot f(v)$ may be practically the same for different targets of a given kind of material (for example, concrete) and R for targets of different strengths may differ only in the factor c. For this reason the constant c has been written explicitly instead of absorbing it in the unknown function $g(x) \cdot f(v)$. This makes possible a future "normalization" of the functions $g(x)$ and $f(v)$ whenever this seems appropriate and desirable.

5/ See page 17 and Table V-B, page 64 in Ref. 1.

6/ See, for example, Ref. 5.

7/ See Ref. 3, especially Appendix B.

8/ See pages 3, 4, 5 in Ref. 1.

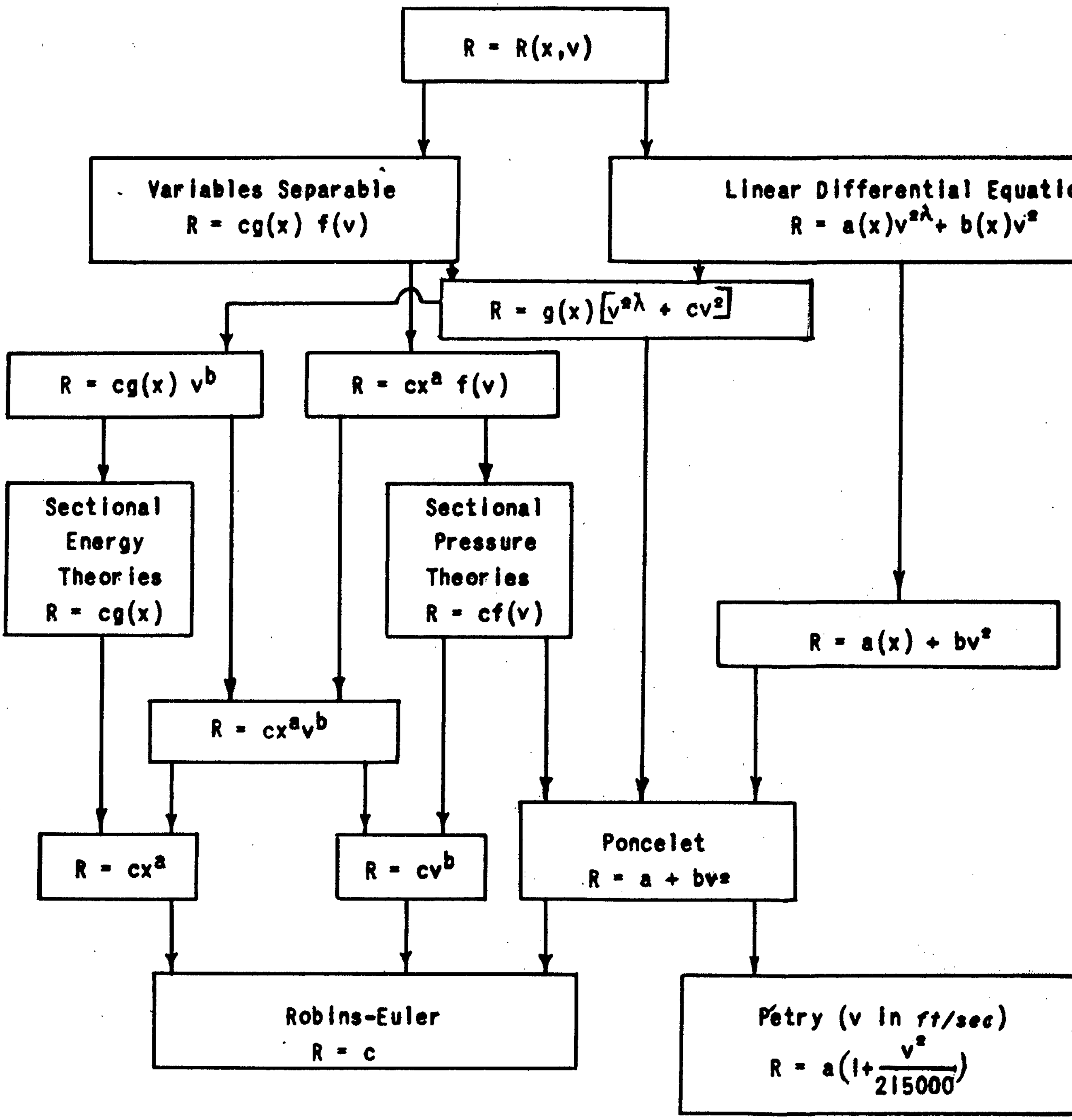
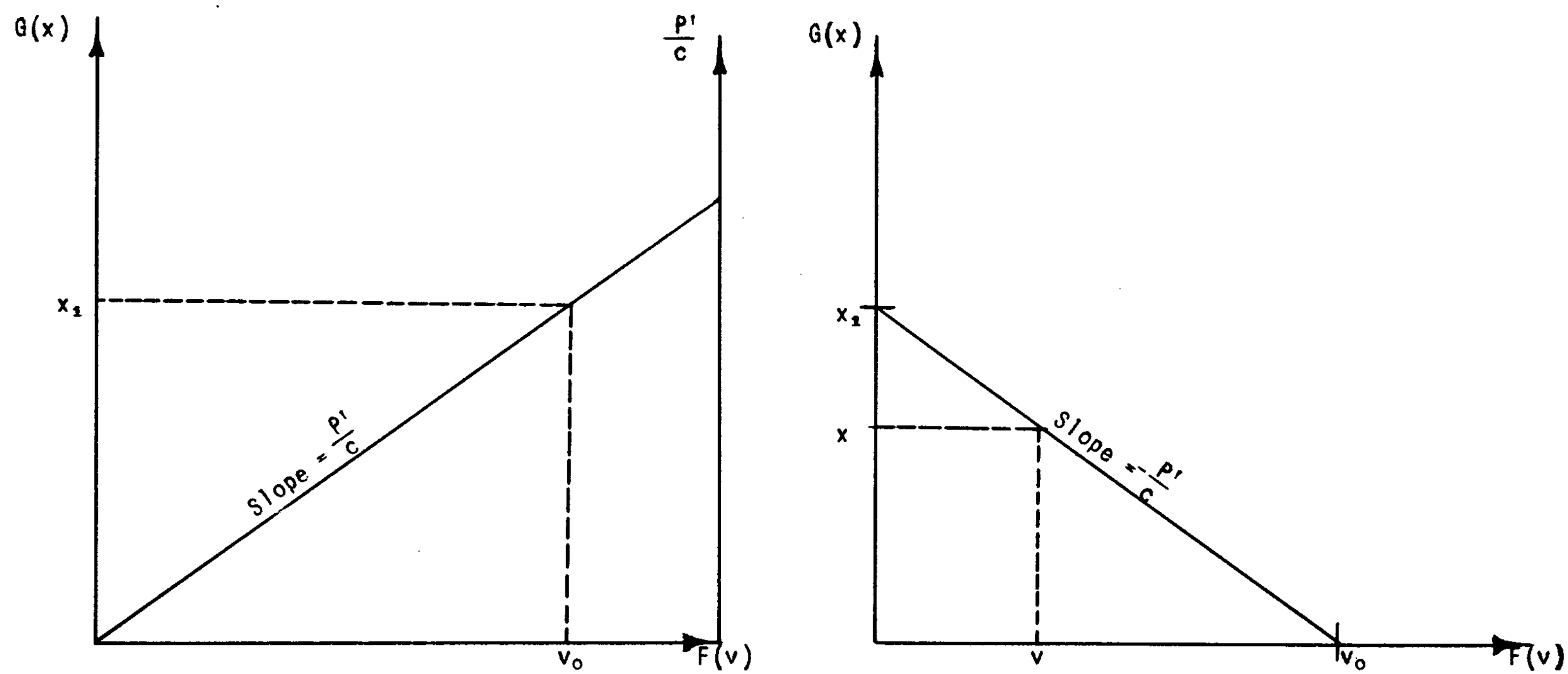


FIGURE 1: CLASSIFICATION OF PENETRATION THEORIES.



(a) $G(x_1) = \frac{P'}{c} F(v_o)$

(b) $\frac{G(x)}{G(x_1)} + \frac{F(v)}{F(v_o)} = 1$

Values for x and v are marked at distances given by G(x) and F(v).
Linear scale for values of $\frac{P'}{c}$
(a) and (b) can be superimposed on same F(v) and G(x) axes.

FIGURE 2: GRAPHICAL REPRESENTATION OF EQUATIONS (14) AND (15) FOR A SEPARABLE FORCE LAW.

Assuming the force law Eq. (8), Eq. (1) may be integrated by separation of the variables x and v. The constant of integration may be determined from either the initial or the final conditions which yield, respectively, the two equivalent solutions:

(10) $$cG(x) = P'[F(v_0) - F(v)],$$

(11) $$c[G(x_1) - G(x)] = P'F(v),$$

where

(12) $$G(x) = \int_0^x g(x)\,dx,$$

(13) $$F(v) = \int_0^v \frac{v\,dv}{f(v)}.$$

The "penetration formula," giving the relation between the striking velocity v_0 and the maximum penetration x_1, is obtained by inserting the final conditions in Eq. (10) or by inserting the initial conditions in Eq. (11):

(14) $$cG(x_1) = P'F(v_0).$$

Dividing Eq. (10) or Eq. (11) by Eq. (14) gives the symmetrical form

(15) $$\frac{G(x)}{G(x_1)} + \frac{F(v)}{F(v_0)} = 1,$$

which is independent of P' and c, for the relation between x and v during penetration. Penetration times are given by a general formula, Eq. (2) or Eq. (3).

Equations (14) and (15) immediately suggest the graphical representation shown in Fig. 2 for the relations between x_1 and v_0, and between x and v, for any separable force law.

The application of the formulae depends on a knowledge of the functions G(x) and F(v). The functional relation between x_1 and v_0 for a given target and projectile can be obtained with reasonable accuracy from penetration experiments. However, this is not sufficient to determine both G(x) and F(v) uniquely. In fact it is evident that one of these can be assumed arbitrarily

Table I. Summary of relations for separable force laws.

Applicable Theory*	$\underline{R}[=c\cdot g(x)\cdot f(v)]$	$G(x)[=\int_0^x g(x)\,dx]$	$F(v)\left[=\int_0^v \frac{v\,dv}{f(v)}\right]$	Relation for time of penetration, t_1, in terms of $K\left[=\frac{v_o t_1}{x_1}\right]$			Relation for $\underline{t}$ during penetration**
	$c\,g(x)\cdot v^b$	—	$\frac{v^{2-b}}{2-b}$	$\frac{1}{x_1}\int_0^{x_1}\frac{dx}{\left[1-\frac{G(x)}{G(x_1)}\right]^{\frac{1}{2-b}}}=K(x_1)$			$v_o t=\int_0^x\frac{dx}{\left[1-\frac{G(x)}{G(x_1)}\right]^{\frac{1}{2-b}}}$ = function of x and x_1
$\underline{E}$	$c\,g(x)$	—	$\frac{v^2}{2}$	$\frac{1}{x_1}\int_0^{x_1}\frac{dx}{\sqrt{1-\frac{G(x)}{G(x_1)}}}=K(x_1)$			$v_o t=\int_0^x\frac{dx}{\sqrt{1-\frac{G(x)}{G(x_1)}}}$ = function of x and x_1
	$c x^a f(v)$	$\frac{x^{1+a}}{1+a}$	—	$\frac{v_o}{(1+a)F(v_o)}\int_0^{v_o}\frac{dv}{f(v)\left[1-\frac{F(v)}{F(v_o)}\right]^{\frac{a}{1+a}}}=K(v_o)$			$\frac{t_1-t}{x_1}=\frac{1}{(1+a)F(v_o)}\int_0^v\frac{dv}{f(v)\left[1-\frac{F(v)}{F(v_o)}\right]^{\frac{a}{1+a}}}$ = function of v and v_o
$\underline{P}$	$c f(v)$	x	—	$\frac{v_o}{F(v_o)}\int_0^{v_o}\frac{dv}{f(v)}=K(v_o)$			$\frac{v(t_1-t)}{x_1-x}=K(v)$ = function of v
	$c x^a v^b$	$\frac{x^{1+a}}{1+a}$	$\frac{v^{2-b}}{2-b}$	$\frac{\Gamma\left(\frac{2+a}{1+a}\right)\cdot\Gamma\left(\frac{1-b}{2-b}\right)}{\Gamma\left(\frac{1}{1+a}+\frac{1-b}{2-b}\right)}$	$\alpha=\frac{2-b}{1+a}$	$x_1=\left(\frac{P'}{c\alpha}\right)^{\frac{1}{1+a}}v_o^{\alpha}$ $K=\frac{v_o t_1}{x_1}$ = constant	$\frac{v_o t}{x_1}=\frac{1}{x_1}\int_0^x\frac{dx}{\left[1-\left(\frac{x}{x_1}\right)^{1+a}\right]^{\frac{1}{2-b}}}$ = function of $\frac{x}{x_1}$
E	$c x^a$	$\frac{x^{1+a}}{1+a}$	$\frac{v^2}{2}$	$\sqrt{\pi}\,\frac{\Gamma\left(\frac{2+a}{1+a}\right)}{\Gamma\left(\frac{1}{1+a}+\frac{1}{2}\right)}=\sqrt{\pi}\,\frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right)}$	$\alpha=\frac{2}{1+a}$		$\frac{v_o t}{x_1}=\frac{1}{x_1}\int_0^x\frac{dx}{\sqrt{1-\left(\frac{x}{x_1}\right)^{1+a}}}$ = function of $\frac{x}{x_1}$
P	$c v^b$ (de Giorgi)	x	$\frac{v^{2-b}}{2-b}$	$\frac{2-b}{1-b}=\frac{\alpha}{\alpha-1}$	$\alpha=2-b$ $a=0$		$\frac{v(t_1-t)}{x_1-x}=K=\frac{\alpha}{\alpha-1}$ = constant
E, P	c Robins-Euler	x	$\frac{v^2}{2}$	2	$\alpha=2$ $a=0$		$\frac{v(t_1-t)}{x_1-x}=K=2$ = constant
P	$a+bv^2$ Poncelet	x	$\frac{1}{2b}\ln\left(1+\frac{b}{a}v^2\right)$	$\frac{2u_o\tan^{-1}u_o}{\ln(1+u_o^2)}=K(v_o)$ with $u_o=v_o\sqrt{\frac{b}{a}}$			$\frac{v(t_1-t)}{x_1-x}=K(v)$ = function of v with $u=v\sqrt{\frac{b}{a}}$
P	$a\left(1+\frac{v^2}{215000}\right)$ Petry	Same as Poncelet case with $\underline{v}$ in ft/sec and $a/b = 215000\ \text{ft}^2/\text{sec}^2$; $\sqrt{\frac{b}{a}}=2.157\times10^{-3}$ sec/ft.						

General Notes:

Penetration formula: $c\cdot G(x_1)=P'\cdot F(v_o)$

Relation between $\underline{x}$ and $\underline{v}$: $\frac{G(x)}{G(x_1)}+\frac{F(v)}{F(v_o)}=1$. Independent of $\underline{P'}$ and $\underline{c}$.

The relations for $\underline{t_1}$ and $\underline{t}$ are independent of $\underline{P'}$ and $\underline{c}$.
Even though $K\left[=\frac{v_o t_1}{x_1}\right]$ is a dimensionless pure number, it does not follow from "model theory" that $\underline{K}$ must have the same constant value for all penetrations.

*$\underline{E}$ denotes sectional energy theories, $R=c\,g(x)$ } See Ref. 2.
$\underline{P}$ denotes sectional pressure theories, $R=c f(v)$ }

**Where $\underline{K}$ appears in this column it is the constant or function defined in the previous column giving the relation for t_1.

and the other then determined to fit the observed penetration curve. The values of $\underline{t}$ and t_1 are, therefore, not uniquely determinable from a knowledge of the penetration curve for a given target and projectile alone.

If, however, the experimental penetration curves connecting x_1 and v_0 are known for a wide enough range of $\underline{P'}$ values, then it is, in principle, possible to test whether the form of Eq. (14) can be fitted to the data with sufficient accuracy, and, if so, to evaluate the functions $G(x)$ and $F(v)$. Specifically, this requires that penetration curves be obtained for different values of $\underline{P'}$ with the same value of $\underline{c}$. The different values of $\underline{P'}$ are obtained by changing the mass of the projectile. The constancy of $\underline{c}$ may be assumed if the target remains the same and if the size and shape of the different projectiles remain constant.

Table I gives a summary of some of the relations that may be derived for the separable force laws shown in Fig. 1. In each case the functions $G(x)$ and $F(v)$ are given insofar as they may be simplified from Eqs. (12) and (13). The penetration formula, Eq. (14), and the relation given by Eq. (15) between $\underline{x}$ and $\underline{v}$ during penetration are not written out in the table for each case since these are completely specified by the functions $G(x)$ and $F(v)$ as given.

The relations given for t_1 and $\underline{t}$ can be worked out from Eqs. (2) and (3). The details of these calculations are not given in order to save space It will be noted that the relations for t_1 and $\underline{t}$ can all be given in a form independent of $\underline{P'}$ and $\underline{c}$. Where $\underline{K}$ appears in the relation for $\underline{t}$ it is the constant or function defined in the previous column giving the relation for t_1. The use of the dimensionless quantity $v_0 t_1/x_1$ has been discussed in previous reports.[9/]

6. The force law $R = cx^a v^b$

The first four lines of Table I contain examples of separable force laws in which one of the functions, $G(x)$ and $F(v)$, still remains arbitrary.

9/ See Fig. 1, page 6 in Ref. 2 and pages 32, 30 in Ref. 3.

In the next four lines the forms of both functions are specified as constant powers of x and v, respectively. The penetration formula resulting from the assumption

$$(16) \qquad R = cx^a v^b \qquad a \geq 0, \qquad b \geq 0$$

is

$$(17) \qquad x_1 = \left(\frac{P'}{\alpha c}\right)^{1/(1+a)} v_0^{\alpha},$$

where

$$(18) \qquad \alpha = \frac{2 - b}{1 + a}$$

and the time of penetration t_1 satisfies the relation

$$(19) \qquad K = \frac{v_0 t_1}{x_1} = \frac{\Gamma\left(\frac{2 + a}{1 + a}\right)\Gamma\left(\frac{1 - b}{2 - b}\right)}{\Gamma\left(\frac{1}{1 + a} + \frac{1 - b}{2 - b}\right)} = \text{constant}.$$

Figure 3 shows the values of K and α obtained from Eqs. (18) and (19) for various values of a and b. Figure 3 covers all four cases of Eq. (16) as given in the fifth, sixth, seventh, and eighth lines of Table I.

For the Robins-Euler theory $K = 2$ and $\alpha = 2$. It is, however, evident from Fig. 3 that $K = v_0 t_1/x_1 = 2$ does not necessarily imply a constant resisting force; in fact, K can be made to have the value 2 for any value of α, by proper choice of a and b.

If K and α are known, a and b are thereby determined, and conversely. Thus, the force law could be evaluated from measurements of v_0, x_1, and t_1 provided that the measurements always gave the same constant values for K and α.

One way of using Fig. 3 may be illustrated by a problem of the following kind.

Assume that the normal penetration x_1 of a nondeforming projectile into concrete is proportional to the 3/2 power of the striking velocity v_0 (that is, $\alpha = 3/2$) and that the resisting force is representable by Eq. (16).

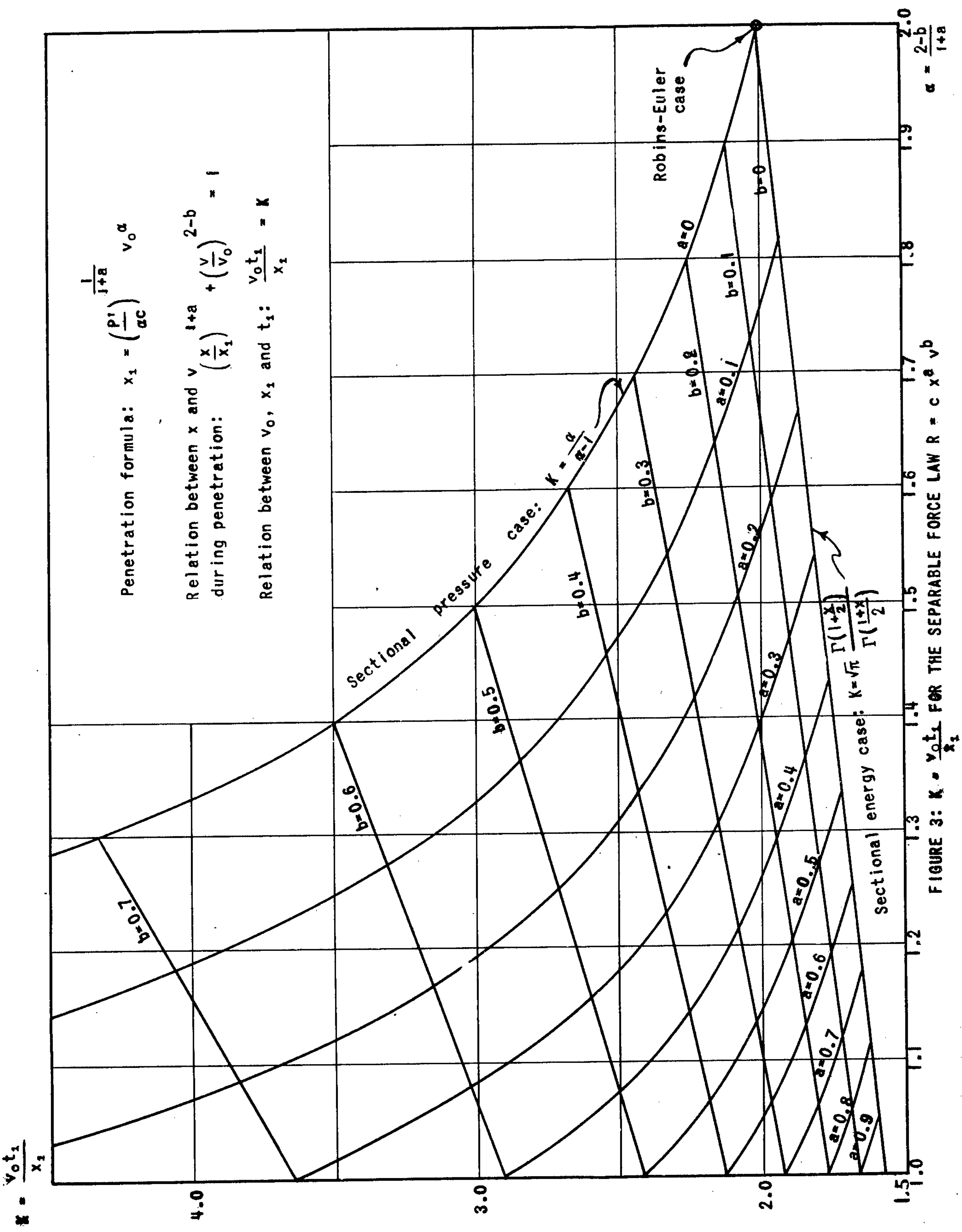


FIGURE 3: $K = \frac{v_o t_1}{x_1}$ FOR THE SEPARABLE FORCE LAW $R = c\,x^a v^b$

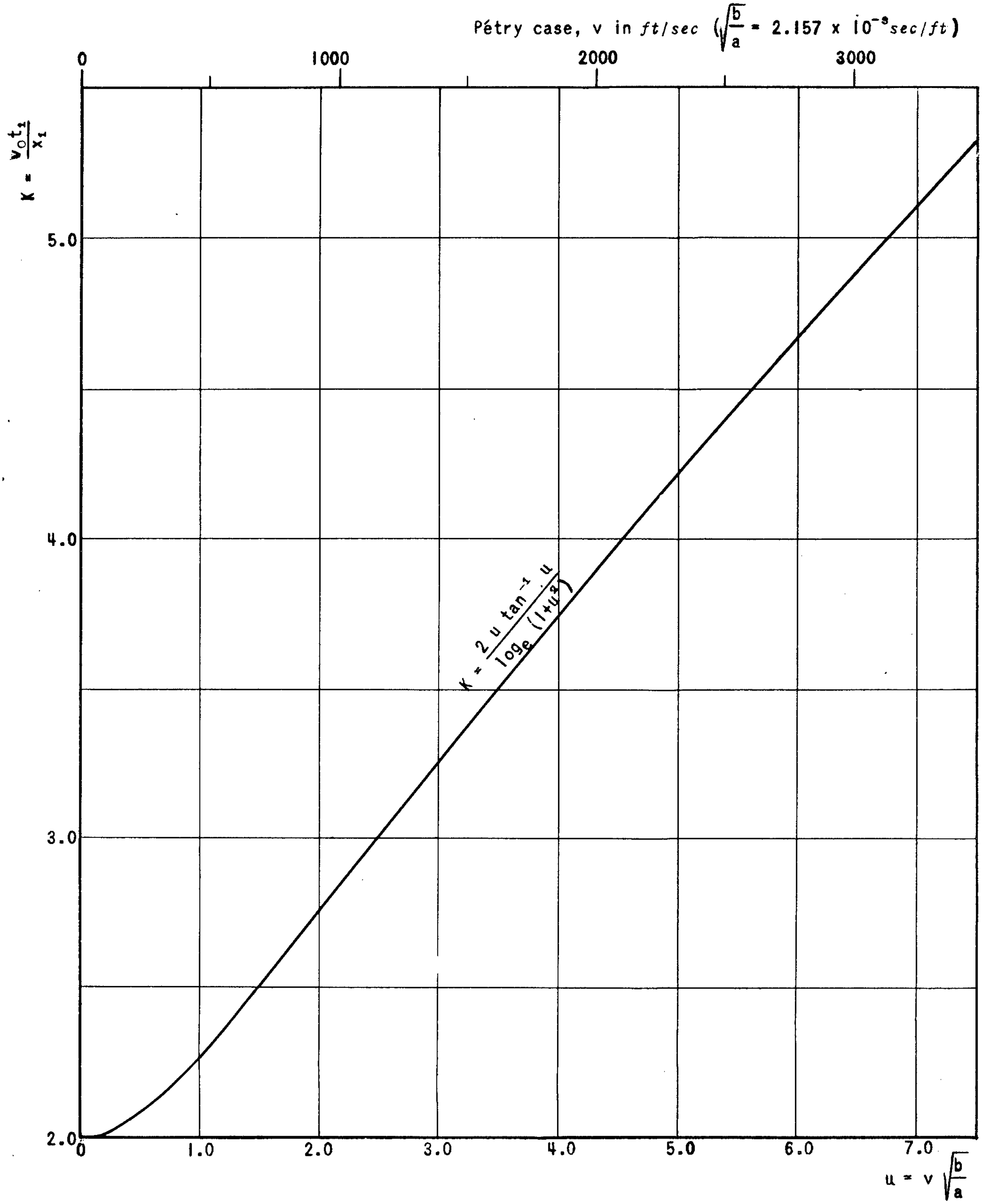


FIGURE 4: $K = \frac{v_0 t_1}{x_1}$ FOR THE PONCELET FORCE LAW.

What is the time of penetration, t_1, of a projectile that penetrates 4.0 ft at a striking velocity of 2000 ft/sec?

We have $t_1 = Kx_1/v_0 = K \times 4/2 \times 10^3 = 2 \times 10^{-3} K$ sec. For $\alpha = 3/2$, Fig. 3 gives $1.8 \leq K \leq 3.0$; hence t_1 lies between 3.6 and 6.0 msec.

If, in addition, it is known that for projectiles of the same caliber and shape the penetration is proportional to the projectile mass for a constant striking velocity, then since P' is the sectional density, from Eq. (17) $a = 0$ and $K = 3.0$, $t_1 = 6.0$ msec. If, however, the penetration is only proportional to the 5/6 power of the projectile mass for constant v_0, then $a = 0.2$, $K = 2.08$, $t_1 = 4.16$ msec.

7. The Poncelet force law $R = a + bv^2$

The last two lines of Table I refer to the Poncelet force law

$$R = a + bv^2 \tag{20}$$

and to the special case thereof which gives the Pétry penetration formula.

Times of penetration can be obtained from the graph of

$$K(v) = \frac{2u \tan^{-1} u}{\ln(1 + u^2)}, \tag{21}$$

where

$$u = v\sqrt{\frac{b}{a}} \tag{22}$$

shown in Fig. 3. In the Pétry case $a/b = 215000$ ft^2/sec^2 10/ and

$$u = 2.157 \times 10^{-3} v,$$

where v is in feet per second as given on the scale at the top of Fig. 4.

One way of using Fig. 4 may be illustrated by a problem similar to the one used in connection with Fig. 3.

Assume that the Pétry force law holds. What is the time of penetration if $x_1 = 4.0$ ft and $v_0 = 2000$ ft/sec?

From Fig. 4 we find $K = 3.89$ and, hence, $t_1 = Kx_1/v_0 = 7.78$ msec.

10/ See page 15 in Ref. 1.

8. Sectional-pressure theories and the time of penetration

If the resisting force R depends only on the instantaneous velocity v during penetration and not on the depth x,

$$R = cf(v); \tag{23}$$

then Eqs. (14) and (11) become, respectively,

$$x_1 = \frac{P'}{c} F(v_o), \tag{24}$$

$$x_1 - x = \frac{P'}{c} F(v), \tag{25}$$

while, from Eq. (3), we have

$$t_1 = \frac{P'}{c} \int_0^{v_o} \frac{dv}{f(v)} \tag{26}$$

and

$$t_1 - t = \frac{P'}{c} \int_0^{v} \frac{dv}{f(v)}. \tag{27}$$

The theories that fall in this class are called sectional-pressure theories since, according to Eq. (24), the penetration x_1 is proportional to the sectional pressure P. They are marked with the letter P at the left side of Table I, and include all of the classical theories -- Eqs. (4), (5), (6), and (7).

According to any sectional-pressure theory with $R = c \cdot f(v)$, the remaining penetration $x_1 - x$ beyond any point x of the path is related to the remaining velocity v at that point in the same way that the total penetration x_1 is related to the striking velocity v_o. [Compare Eqs. (24) and (25).][11] Furthermore, the remaining time $t_1 - t$ is related to v in the same way that the total time t_1 is related to v_o. [Compare Eqs. (26) and (27).] Thus we have from Eqs. (24) and (26),

$$\frac{v_o t_1}{x_1} = \frac{v_o}{F(v_o)} \int_0^{v_o} \frac{dv}{f(v)} = K(v_o), \tag{28}$$

11/ The case $F(v) = v^{3/2}$, as discussed in Ref. 5, is a particular example of this general statement.

and, from Eqs. (25) and (27),

$$\frac{v(t_1 - t)}{x_1 - x} = \frac{v}{F(v)} \int_0^v \frac{dv}{f(v)} = K(v) \tag{29}$$

as shown in Table I. The values of $\underline{P'}$ and $\underline{c}$ have been eliminated in obtaining these expressions, and $\underline{K}$ is the same function of its argument in both cases.

The time $\underline{t}$ at any point during penetration may be computed from these equations for any assumed sectional-pressure theory from given values of x_1, v_o, and either $\underline{x}$ or $\underline{v}$ to specify the point at which $\underline{t}$ is to be found. The remaining one of the pair $\underline{x},\underline{v}$ is first obtained from Eq. (15), which on a sectional-pressure theory becomes

$$\frac{x}{x_1} + \frac{F(v)}{F(v_o)} = 1. \tag{30}$$

Then t_1 and $\underline{t}$ can be evaluated from Eqs. (28) and (29). These steps can be carried out with the aid of two graphs constructed for the particular $f(v)$ assumed in Eq. (23); namely, a graph of Eq. (30) as suggested in Fig. 2(b), and a graph of $K(v)$ as illustrated in Fig. 4.

This method of computation may be illustrated by the following examples.

Assume (as before) $v_o = 2000$ ft/sec and $x_1 = 4.0$ ft. Find the remaining velocity $\underline{v}$ and the time $\underline{t}$ at the depth $x = 3.0$ ft for each of the assumptions:

A: $f(v) = \sqrt{v}$,

B: $f(v) = 1 + \dfrac{v^2}{215000}$. (Pétry)

<u>For A</u> we have

$$F(v) = \int_0^v \frac{v\,dv}{f(v)} = \frac{2}{3} v^{3/2}, \qquad \alpha = 3/2,$$

$$\frac{x}{x_1} + \left(\frac{v}{v_o}\right)^{3/2} = 1, \quad \text{or} \quad v = 794 \text{ ft/sec},$$

and

$$K = \frac{\alpha}{\alpha - 1} = 3.0 = \frac{v_o t_1}{x_1} = \frac{v(t_1 - t)}{x_1 - x};$$

whence

$$t_1 = 3.0x_1/v_0 = 6.0 \text{ msec},$$
$$t_1 - t = 3.0(x_1 - x)/v = 3.0/794 = 3.78 \text{ msec},$$
$$t = 2.22 \text{ msec}.$$

For B we have the Pétry case

$$\frac{x}{x_1} + \frac{\ln\left(1 + \frac{v^2}{215\,000}\right)}{\ln\left(1 + \frac{v_0^2}{215\,000}\right)} = 1, \quad \text{or} \quad v = 487 \text{ ft/sec}.$$

From Fig. 4,

$$K = 3.89 \quad \text{for} \quad v = 2000 \text{ ft/sec},$$
$$K = 2.28 \quad \text{for} \quad v = 487 \text{ ft/sec};$$

whence

$$t_1 = 3.89x_1/v_0 = 7.78 \text{ msec},$$
$$t_1 - t = 2.28(x_1 - x)/v = \frac{2.28}{487} = 4.68 \text{ msec},$$
$$t = 3.10 \text{ msec}.$$

9. A perforation hypothesis

The analysis so far has dealt only with the penetration of a massive target by a nondeforming projectile. If the target is thin enough the projectile will perforate the target; that is, the projectile will pass through the target and will emerge from the back face with a residual velocity v_r. For a target of thickness e, there will be a limit velocity v_ℓ for which the projectile will just perforate the target.

If we assume that the motion during perforation is governed by a separable force law,

(31) $$R = c \cdot g_e(x) \cdot f(v),$$

then

(32) $$g_e(x) = 0 \quad \text{for} \quad x \geq x_e$$

since the force must fall to zero at the end of perforation. The distance x_e over which target resistance acts on the projectile will be roughly equal to the target thickness e. However, in the case of a brittle, scab-forming material like concrete we may have $x_e < e$, while for cohesive or ductile materials like steel we may have $x_e > e$.

Integrating the equation of motion, Eq. (1), for the separable force law, Eq. (31), in the case of perforation ($v_o \geq v_\ell$, $v_r \geq 0$) we get

$$P'[F(v_o) - F(v_r)] = cG_e, \tag{33}$$

where

$$G_e = \int_o^{x_e} g_e(x)\,dx = \text{constant}. \tag{34}$$

Since by definition of v_ℓ, $v_\ell = v_o$ when $v_r = 0$, we obtain from Eq. (33)

$$\frac{cG_e}{P'} = F(v_\ell) = F(v_o) - F(v_r). \tag{35}$$

It would be natural to assume that the function f(v) in Eq. (31) for perforation is the same as f(v) in Eq. (8) for penetration when the targets are of the same material. Then the relation given by Eq. (35) among v_ℓ, v_o, and v_r involves the same F(v) as defined for penetration in Eq. (13). Thus perforation measurements of striking and residual velocities would offer a means of evaluating the function F(v). Alternatively, if F(v) is known from penetration data, Eq. (35) offers a method of estimating residual velocities v_r.

10. Conclusions

Of the two general cases of force laws that can be explicitly integrated, the separable force laws, Eq. (8), lead to simpler general expressions than the generalized Poncelet force laws, Eq. (9),[12] especially when x, v, t, and t_1 are to be calculated.

For separable force laws the relation between x and v during penetration can be written in the form given by Eq. (15) in which P' and c do not appear, but in which both x_1 and v_o are used explicitly. The examples given illustrate the fact that this leads to simple methods for computing x, v, t, and t_1.

12/ Compare Ref. 3.

The resisting pressure R has been defined as the resisting force divided by A, the maximum cross-sectional area of the projectile. It is, therefore, to be expected that R will depend on x at least to the extent that the "amplitude of impression"[13] increases with the depth x of nose penetration while the pointed nose of a projectile or bomb is entering the target. On this basis a pure sectional-pressure theory, involving no dependence of R on x, would not be correct. In a qualitative way Fig. 2 shows the large effect that a small degree of x-dependence may have on the computed times of penetration, t_1. It has been pointed out[14] that sectional-pressure theories with $df/dv > 0$ always lead to values of $K = v_0 t_1/x_1$ larger than 2, while for sectional-energy theories with $dg/dx > 0$, K is less than 2. It is, therefore, felt that pure sectional-pressure theories (in particular, the classical theories) tend to give values of t_1 that are too large, and that better values may often be gotten by the crude but simple assumption that $K = 2$. For concrete, K may increase somewhat with x_1.[15]

List of References

1. "Terminal ballistics I," by H. P. Robertson, Committee on Passive Protection Against Bombing, Interim Report, Jan. 1941. Unclassified.

2. "Penetration theory: Estimates of velocity and time during penetration," by R. A. Beth, OTB-7 (OSRD-4720). Confidential.

3. "Concrete penetration," by R. A. Beth, NDRC Report A-319 (OSRD-4856), March 1945. Confidential.

4. "An electromagnetic method for measuring projectile velocity during penetration," by R. A. Beth and E. J. Schaefer, NDRC Report A-329 (OSRD-5175).

5. "Penetration of projectiles into concrete," REN 376, British Ministry of Home Security, June 1944. British Secret.

13/ See page 22, Ref. 1.

14/ See Fig. 1 in Ref. 2.

15/ Compare Fig. 11, page 32 in Ref. 3.

Office of Scientific Research and Development

NATIONAL DEFENSE RESEARCH COMMITTEE

Technical Reports Section, Vice Chairman's Office

Room 4724, Munitions Building, Washington 25, D.C.

JUL 7 1945

TO: ~~[illegible]~~

~~Att: [illegible] Oswald [illegible]~~

FROM: Hannah M. Diamond, National Defense Research Committee

Room 4724, Munitions Building, Washington 25, D.C.

Transmitted herewith are the items listed below. Please sign and return this receipt immediately. The copy may be retained by you.

NDRC Number	OSRD Number	Copy Number
[illegible]-333	5258	5[illegible]

I have personally received the material described above and accept full responsibility for its safe handling, storage, and transmittal elsewhere in accordance with existing regulations of OSRD, the War Department, or the Navy Department, whichever is pertinent.

Date: 7-11-45 ________________ (Signature)

~~CONFIDENTIAL~~